# Comparing energy loss and $p_\perp$-broadening in perturbative QCD with strong coupling $\mathcal{N} = 4$ SYM theory*

Fabio Dominguez,[1,†] C. Marquet‡,[2,1,§] A. H. Mueller,[1,¶] Bin Wu**,[3,1,††] and Bo-Wen Xiao[1,‡‡]

[1] *Department of Physics, Columbia University, New York, NY, 10027, USA*

[2] *Service de Physique Théorique, CEA/Saclay, 91191 Gif-sur-Yvette cedex, France*

[3] *Department of Physics, Peking University, Beijing, 100871, P.R. China*

(Dated: July 11, 2008)

We compare medium induced energy loss and $p_\perp$-broadening in perturbative QCD with that of the trailing string picture of SYM theory. We consider finite and infinite extent matter as well as relativistic heavy quarks which correspond to those being produced in the medium or external to it. When expressed in terms of the appropriate saturation momentum, we find identical parametric forms for energy loss in perturbative QCD and SYM theory. We find simple correspondences between $p_\perp$-broadening in QCD and in SYM theory although $p_\perp$-broadening is radiation dominated in SYM theory and multiple scattering dominated in perturbative QCD.

PACS numbers: 12.38.Mh, 11.25.Tq, 25.75.-q

* This work is supported in part by the US Department of Energy.

‡ C.M. is supported by the European Commission under the FP6 program.

** B.W. is supported by China Scholarship Council.

†Electronic address: fad2111@columbia.edu

§Electronic address: cyrille@phys.columbia.edu

¶Electronic address: amh@phys.columbia.edu

††Electronic address: bw2246@columbia.edu

‡‡Electronic address: bowen@phys.columbia.edu

## I. INTRODUCTION

The purpose of this paper is to compare energy loss and $p_\perp$-broadening for relativistic heavy quarks passing through a hot perturbative QCD plasma with that of the trailing string description[1, 2] for a hot SYM plasma. We shall see that for energy loss in infinite extent matter the formulas for perturbative QCD and for a strongly coupled $\mathcal{N} = 4$ SYM theory are parametrically identical when expressed in terms of the relevant saturation momentum, $Q_s$. For $p_\perp$-broadening the SYM result is parametrically identical to a (subdominant) perturbative QCD term which in the strong coupling limit of SYM theory becomes the leading term. This comparison in infinite matter is then extended to finite extent matter where the trailing string picture has not yet been developed. We find results identical with infinite matter when expressed in terms of $Q_s$. We have not attempted comparison with a competitive theory of energy loss [3] based on a Wilson loop calculation because that approach is essentially perturbative with the SYM theory only being used to give an evaluation of the transport coefficient, $\hat{q}$.

In this introduction we motivate and summarize our results. In Sec. II, these results are described in more detail with additional details and supporting evidence for our picture given in Sec. III, IV and V.

We begin by recalling the picture of energy loss for a relativistic heavy quark[4] traveling through an infinite extent hot perturbative QCD plasma. We suppose the heavy quark to have rapidity $\eta$ at the time over which we focus. The heavy quark has a cloud of gluons, labeled by energy $\omega$ and transverse momentum $k_\perp$. Energy loss is dominated by the maximum energy gluons which are freed, or radiated, in the plasma[5, 6, 7, 8]. Gluons are freed if their transverse momentum is less than the saturation momentum corresponding to the gluon coherence time, $t_c$. This immediately gives

$$-\frac{dE}{dt} \propto \alpha N_c \frac{(\omega)_{max}}{t_c} = \alpha N_c Q_s^2, \tag{1}$$

where we have used $t_c = \frac{(\omega)_{max}}{Q_s^2}$ as the relevant coherence time. The condition $\frac{(\omega)_{max}}{Q_s} = \cosh\eta$ as well as $Q_s^2 = \hat{q} t_c$ converts Eq. (1) to a more standard form. The $\alpha N_c$ in Eq. (1) is just the probability of having a gluon, in the kinematic regime in question, in the heavy quark wavefunction.

We now note that Eq. (1) is also true for a strong coupling $\mathcal{N} = 4$ SYM plasma if the

replacement $\alpha N_c \to \frac{\sqrt{\lambda}}{2\pi}$ is made, where $\lambda = g_{YM}^2 N_c$, and if the saturation momentum is now taken as $Q_s^2 \propto (t_c T^2)^2$ as found in Ref. [9]. (The stronger dependence of $Q_s^2$ on $t_c$ in SYM theory, as compared to perturbative QCD, comes about because high energy scattering in SYM theory is dominated by a $J \simeq 2$ singularity rather than the $J \simeq 1$ singularity in perturbative QCD.) At first sight it may seem strange that a perturbative picture, giving Eq. (1), could also lead to a correct result for a strongly coupled $\mathcal{N} = 4$ SYM theory. However, we know that for a static heavy quark in the vacuum the distribution of energy in the fields surrounding the center of the heavy quark is the same for strong coupling SYM theory and for QCD, or even QED[10, 11]. As we show in Sec. III, this leads to an energy distribution in momentum space for a high energy heavy quark in SYM theory exactly as in QCD when one replaces $\alpha N_c$ by $\frac{\sqrt{\lambda}}{2\pi}$. The saturation momentum then determines the scale at which quanta in the heavy quark wavefunction are freed in the plasma, and this is the only dynamics necessary to get the parametric form Eq. (1).

Further evidence for this picture comes from $p_\perp$-broadening of a heavy quark in infinite extent matter. In QCD this broadening occurs mainly by direct multiple scattering of the heavy quark by thermal quanta[12, 13]. There is a subdominant effect, down by $\alpha N_c$, where the heavy quark gets increased transverse momentum by radiation of gluons. However, in strong coupling SYM theory $\alpha N_c$ is replaced by $\frac{\sqrt{\lambda}}{2\pi}$ and now radiation becomes dominant. Thus by simply allowing the emissions which give the energy loss to have a random transverse momentum, whose size is fixed to be that of the saturation momentum, one arrives at a dominant form for $p_\perp$-broadening in $\mathcal{N} = 4$ SYM theory (see Eq. (22)) which is the same as that for radiative broadening in QCD and which agrees with calculations done using the trailing string picture.

Calculations of small oscillations on the trailing string[14, 15] lead one to identifying a point on the string which corresponds to a horizon on the metric induced on the worldsheet of the trailing string as it moves through the QCD plasma. This horizon corresponds to a $u-$value (see the metric Eq. (9)) equal to $u_s = \pi T \sqrt{\cosh \eta}$ which is the same as the value of the saturation momentum for the heavy quark in the plasma. In our picture all quanta in the heavy quark having transverse momenta less than $Q_s$ are freed while those having transverse momenta greater than $Q_s$ remain part of the heavy quark. In Sec. V, we verify that the part of the trailing string having $u > u_s$ corresponds to energy which spatially lies very close to the center of the heavy quark while the $u < u_s$ part of the trailing string

corresponds to energy lagging too far behind the heavy quark center to be considered part of it. This part of the trailing string corresponds to waves in the plasma, and this matches our interpretation in terms of quanta which have been freed from the wavefunction of the heavy quark. We also note that $u_s$ is the same point as that identified in Ref. [16] as separating branching in the vacuum, $u > u_s$, from branching in the medium, $u < u_s$. In the trailing string picture this agrees with our interpretation of the $u > u_s$ part of the string as being the same as that of a heavy quark in the vacuum while the $u < u_s$ part of the string is interpreted as quanta separate from the heavy quark.

Energy loss in finite matter is more subtle in both QCD and in $\mathcal{N} = 4$ SYM theory. The first issue that arises is whether a heavy quark going through a finite extent of matter is "bare" or "dressed"[5, 6]. If the heavy quark does not have too large a rapidity, $\cosh\eta \ll (LT)^2$ in a SYM plasma, then the matter is effectively infinite extent and we are back to our earlier discussion. However, if $\cosh\eta$ is large there is a strong distinction between bare and dressed heavy quarks. A dressed heavy quark is one that has its gluon cloud completely developed as it enters the medium. In the trailing string picture a vertical string going from $u = 0$ to $u = u_m$ entering the medium traveling at rapidity $\eta$ corresponds to a dressed quark entering the medium. For a dressed heavy quark energy loss simply corresponds to freeing those quanta having transverse momentum less than $Q_s$, with the length determining $Q_s$ being given by the length $L$ of the matter. In the case of the trailing string it, parametrically, corresponds to freeing all the string lying below $u = u_s$. The result, Eq. (29), takes the same form for a QCD plasma and for a SYM plasma.

For bare quarks we are faced with a more serious challenge. In QCD one can simply produce a heavy quark-antiquark pair in a hard collision and then allow, say, the heavy quark to go through the medium. In SYM theory consider a heavy quark-antiquark pair initially at rest and very close together. We then rapidly accelerate the quark and antiquark in opposite directions using an external electric field. When the quark has reached the desired rapidity we turn off the electric field and let the quark continue to pass through the medium. We have found an exact solution for the motion of the string during the period of acceleration, and this solution is described in Sec. IV and in Ref. [17]. One can see from this solution, and the discussion in Sec. II, that now the heavy quark is missing much of its cloud of quanta as it passes through the medium. The key to determining medium induced energy loss is an understanding of the time at which components of the final dressed heavy quark are

formed. In QCD this is well understood. In Sec. II we motivate, but ultimately conjecture that the formation times are the same for strong coupling SYM theory and perturbative QCD. This leads to an energy loss formula as in Eq. (1) for QCD and the same, with $\alpha N_c \to \frac{\sqrt{\lambda}}{2\pi}$, for SYM theory but where now $Q_s^2 = \hat{q}L$ in perturbative QCD and $Q_s^2 \sim (LT^2)^2$ for hot SYM matter. We note in passing that the induced energy loss for hot SYM matter goes as the cube of the length of the material in contrast to a quadratic law in QCD[18] while the total $p_\perp^2$ picked up traversing the medium goes like $L^2$ for hot SYM matter instead of the linear dependence on $L$ in hot perturbative QCD[12, 13].

## II. COMPARISON BETWEEN QCD AND SYM RESULTS FOR ENERGY LOSS AND $p_\perp$-BROADENING

In this section we give a detailed comparison of results on energy loss and $p_\perp$-broadening for a heavy quark in QCD with corresponding results in $\mathcal{N} = 4$ SYM theory. Although the results for the strong coupling SYM theory have been derived (In some cases the derivations will be given for the first time in later section of this paper.) using the AdS/CFT correspondence[19, 20, 21] with the SYM theory, we here emphasize the comparison of QCD directly with the SYM theory. We rely on explicit calculations on the string theory side of the correspondence to give confirmation of the basic validity of our picture. We start with energy loss in infinite hot matter where both the QCD and SYM theory results are well known. What we are here noting is that when expressed in terms of the relevant saturation momentum, the results for QCD and SYM are essentially identical for energy loss, however, the physical picture for $p_\perp$-broadening differs in these two theories.

### A. QCD energy loss of heavy quarks in infinite extent hot matter

Consider a heavy quark of mass $M$ passing through hot matter of temperature $T$. We suppose $T/M \ll 1$. Suppose at some time the heavy quark is moving at rapidity $\eta$ so that its energy is $M \cosh\eta$. We also assume, for reasons to be explained below, that $\cosh\eta \ll M^3/\hat{q}$ with the $\hat{q}$ the transport coefficient of the plasma.

The dominant cause of energy loss of the heavy quark is gluon radiation[4, 5, 6, 7, 8] induced by the medium, at least when $\cosh\eta \gg 1$ which is the region with which we

shall be concerned. Induced gluon radiation is caused by gluons in the wavefunction of the heavy quark being freed by interaction with the medium. Suppose a gluon in the quark's wavefunction has energy $\omega$ and transverse momentum $k_\perp$. Such a gluon will be radiated if $k_\perp \leq Q_s$ with $Q_s$ the saturation momentum of the medium corresponding to a length given by the coherence time, $t_c$, of the gluon. The energy loss will be dominated by gluons having $k_\perp$, and $\omega$, as large as possible. Thus, setting $k_\perp = Q_s$, we have

$$\hat{q} t_c = Q_s^2 \tag{2}$$

and using

$$t_c = \frac{\omega}{Q_s^2}, \tag{3}$$

gives

$$Q_s^4 = \hat{q}\omega. \tag{4}$$

Gluons having $\frac{\omega}{k_\perp} = \frac{\omega}{Q_s} > \cosh\eta$ are strongly suppressed in the heavy quark wavefunction[4]. Taking $\omega = Q_s \cosh\eta$ will give the maximum energy and, combined with Eq. (4), gives

$$Q_s^3 = \hat{q}\cosh\eta \tag{5}$$

as the relevant saturation momentum. Now we see why it is necessary to keep $\cosh\eta \ll M^3/\hat{q}$, because this is the condition that $\frac{k_\perp^2}{M^2} = \frac{Q_s^2}{M^2} \ll 1$, which condition is the essential requirement separating heavy quark from light quark radiation dynamics. Now it is a straightforward task to give a parametric form of the rate of energy loss of a heavy quark passing through a plasma as

$$-\frac{dE}{dt} \propto \alpha_s N_c \frac{\omega}{t_c}. \tag{6}$$

The $\alpha_s N_c$ in Eq. (6) corresponds to the number of gluons, in the relevant kinematic domain, in the quark wavefunction while $t_c$ corresponds to the time over which the gluon energy, $\omega$, is emitted. Using Eq. (3) gives

$$-\frac{dE}{dt} \propto \alpha_s N_c Q_s^2. \tag{7}$$

where, $Q_s$ should be evaluated using Eq. (5).

The difference between heavy and light quarks lies not in changing Eq. (7), which remains valid for light quarks, but in Eq. (5) which for light quarks energy loss becomes

$$Q_s^2 \simeq \sqrt{E\hat{q}} \tag{8}$$

with $E$ the energy of the light quark.

### B. Energy loss of a heavy quark in SYM theory in infinite extent hot matter

The problem of energy loss for heavy quarks in a $\mathcal{N} = 4$ SYM plasma has been well studied[1, 2]. Using the AdS/CFT correspondence allows one to evaluate the energy loss in terms of the energy flowing toward the horizon through a string which goes from a $D7$ brane, whose position is given by the heavy quark mass, toward the horizon of the metric. The metric in the $AdS_5$ space can be written as

$$ds^2 = R^2u^2\left[-f\left(u\right)dt^2 + d\vec{x}^2\right] + \frac{du^2R^2}{u^2f\left(u\right)}, \tag{9}$$

with

$$f\left(u\right) = 1 - \left(\frac{u_h}{u}\right)^4 \tag{10}$$

and where $u_h = \pi T$ and $T$ is the temperature of the plasma. The $D7$ brane is located at a position $u_m$ which is related to the heavy quark's mass by

$$M = \frac{\sqrt{\lambda}u_m}{2\pi} \tag{11}$$

corresponding to a string at rest in the vacuum having length $u_m$, as it falls straight down from $u_m$ to $u = 0$ in the fifth dimension. The energy density of the string is

$$\frac{dE}{du} = \frac{\sqrt{\lambda}}{2\pi}. \tag{12}$$

In Eqs. (11) and (12), $\lambda$ is given by $\lambda = g_{YM}^2N_c$. In the energy loss problem one keeps the string moving at a constant rapidity $\eta$ where one imagines an external constant "electric" field acting on the end of the string on the $D7$ brane and furnishing the force necessary to keep the string moving at constant velocity $v$ where $\cosh\eta = 1/\sqrt{1-v^2}$. The resulting rate of energy loss, that is the rate of work done by the external electric field, is[1, 2]

$$-\frac{dE}{dt} = \frac{\pi\sqrt{\lambda}}{2}T^2v^2\cosh\eta \simeq \frac{\pi\sqrt{\lambda}}{2}T^2\cosh\eta \tag{13}$$

where we have only consider the case $\cosh\eta \gg 1$.

We are now going to show that Eq. (13) agrees with Eq. (7) if one uses the saturation momentum appropriate to the $\mathcal{N}=4$ SYM plasma instead of Eq. (5). To get the replacement of Eq. (5) for the SYM plasma, we recall that the saturation momentum corresponding to a length $L$ of such a plasma is[9]

$$Q_s\,(L) \sim LT^2. \tag{14}$$

Using $t_c$ in Eq. (3) as the relevant length , just as we used for calculating the energy loss in QCD, and taking $\omega/Q_s = \cosh\eta$, one gets

$$Q_s^2 \sim T^2 \cosh\eta \tag{15}$$

from Eq. (14). Using Eq. (15) in Eq. (13), one arrives at

$$-\frac{dE}{dt} \sim \sqrt{\lambda} Q_s^2. \tag{16}$$

Except for the replacement of $\alpha N_c$ by $\sqrt{\lambda}$ Eq. (7) and Eq. (16) are the same. The replacement of $\alpha N_c$ by $\sqrt{\lambda}$ is to be expected and already occurs in comparing the static energy density, the electric field squared, of a heavy quark in QCD with that of a heavy quark in $\mathcal{N}=4$ SYM theory. As we show in Sec. III, this naturally leads a gluon distribution in a heavy quark to be the same in QCD and in $\mathcal{N}=4$ SYM theory except for the replacement of $\alpha N_c$ by $\sqrt{\lambda}$. Our picture of energy loss of a heavy quark in an infinite plasma is thus essentially identical in QCD and in $\mathcal{N}=4$ SYM. We do note, however, that we do not have control of constant factors in our description of energy loss in a $\mathcal{N}=4$ SYM plasma.

### C. Transverse momentum broadening of a heavy quark in a QCD plasma

The picture of transverse momentum broadening of a high energy quark in QCD is very simple and is the same for light quarks as for heavy quarks[12, 13]. As the quark passes through the plasma, it interacts with quanta of the plasma through multiple single gluon exchange, each exchange giving a random transverse momentum to the quark, and two gluon exchanges, necessary to keep probability conservation. The formula

$$\frac{dp_\perp^2}{dt} = \hat{q}. \tag{17}$$

gives the rate of increase in the typical $p_\perp^2$ that the quark picks up in passing through the medium in terms of the transport coefficient of medium. One can also write this relation as

$$\frac{dp_\perp^2}{dt} = \frac{dQ_s^2\left(t\right)}{dt}. \tag{18}$$

where $Q_s\left(t\right)$ is the saturation momentum of a length, $t$, of the medium where we do not distinguish between time and length intervals for our relativistic heavy quark. The result given in Eq. (17) and Eq. (18) comes completely from random multiple scattering of the quark in the medium. There is also a contribution from emission of gluons stimulated by the medium. The same gluon emissions which give the energy loss indicated in Eq. (7) naturally give a $p_\perp$-broadening

$$\left(\frac{dp_\perp^2}{dt}\right)_{\text{radiation}} \sim \alpha_s N_c \frac{dQ_s^2\left(t\right)}{dt}. \tag{19}$$

Eq. (19) is perhaps obvious once one writes Eq. (7) in the form

$$-\frac{dE}{dt} \sim \alpha_s N_c \frac{d\omega\left(t\right)}{dt} \tag{20}$$

where $\omega\left(t\right)$ is given by Eq. (4) with the $t$ -dependence of $Q_s$ given by Eq. (2), identifying $t_c$ with $t$. Then Eq. (19) and Eq. (20) directly give the energy loss and transverse momentum broadening due to gluon radiation stimulated by the medium. The contribution of Eq. (19) is usually neglected because it is parametrically smaller than the contribution given in Eq. (18) from multiple scattering. However, as we shall see below in a SYM plasma, transverse momentum broadening of a heavy quark is dominated by gluon radiation because $\alpha N_c$ will be replaced by the large parameter, $\sqrt{\lambda}$.

### D. Transverse momentum broadening of a heavy quark in an infinite extent SYM plasma

The calculation of transverse momentum broadening in a SYM plasma is done in terms of fluctuations on the corresponding string on the gravity side of the AdS/CFT correspondence. The result is[14, 15]

$$\frac{dp_\perp^2}{dt} = 2\pi\sqrt{\lambda}T^3\sqrt{\cosh\eta}. \tag{21}$$

If we identify the saturation momentum $Q_s$ as in Eq. (15), then one can write

$$\frac{dp_\perp^2}{dt} \sim \sqrt{\lambda}T^2 Q_s. \tag{22}$$

or, alternatively, as

$$\frac{dp_\perp^2}{dt} \sim \sqrt{\lambda}\frac{dQ_s^2\left(t\right)}{dt}. \tag{23}$$

where, again, we identify $t$ with $t_c$ the coherence time of the gluons whose emission dominates both the energy loss and transverse momentum broadening of the heavy quark. Eq. (23) is identical, after $\alpha N_c \leftrightarrow \sqrt{\lambda}$, with Eq. (19) adding confirmation to our picture of energy loss and $p_\perp$-broadening as due to gluon emission in the SYM plasma. Once one writes formulas in terms of the relevant saturation momentum, the SYM and QCD pictures become identical. Of course, in the case of $p_\perp$-broadening what is a subdominant effect in QCD, gluon emission, becomes dominant in the SYM plasma.

### E. The saturation momentum and the trailing string

Our picture of infinite extent plasmas has been that a heavy quark loses energy and gets transverse momentum broadening by the emission of gluons having transverse momentum equal to the saturation momentum of the medium corresponding to a medium length given by the coherence length of the emitted gluons which carry the same rapidity as the parent heavy quark. Now we are going to identify this picture more closely with the trailing string picture.

In the trailing string picture, a heavy quark moving at constant rapidity through a SYM plasma correponds to a string moving at the same rapidity in the background metric, Eq. (9) on the AdS side of the correspondence. The shape of the string is[1, 2]

$$z\left(u\right) = z_0 + vt + \frac{v}{2u_h}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{u}{u_h}\right) - \coth^{-1}\left(\frac{u}{u_h}\right)\right] \tag{24}$$

where

$$u_h < u < u_m = \frac{2\pi M}{\sqrt{\lambda}} \tag{25}$$

with $M$ the mass of the heavy quark. In our picture, the parts of the string having $u_m > u > Q_s$ correspond to gluons actually in the heavy quark while those parts of the string having $u_h < u < Q_s$ correspond to freed matter which is no longer part of the heavy quark. The flow of energy past the point

$$Q_s = u_s \equiv \pi T\sqrt{\cosh\eta}, \tag{26}$$

with $Q_s$ given as in Eq. (15), corresponds to the freeing of gluons from the heavy quark and into QCD plasma. Let us now see that $u \sim u_s$ is a natural point on the string for separating the heavy quark from waves in the plasma. Let $\Delta z\,(u)$ be the distance between a point on the trailing string and a corresponding point on a string, having the same motion at $u = u_m$, moving in the vacuum. Using Eq. (24) for $u$ large, one easily finds

$$\Delta z\,(u) = -\frac{v u_h^2}{3}\left(\frac{1}{u^3} - \frac{1}{u_m^3}\right). \tag{27}$$

Now we expect that the range of z-values allowed in a heavy quark moving at velocity $v$ in the vacuum is given by

$$\delta z\,(u) \simeq -\frac{1}{u}\frac{1}{\cosh\eta}, \tag{28}$$

where the first factor on the right hand side of Eq. (28) is the infrared-ultraviolet correspondence while the second factor is due to Lorentz contraction. Those region of $u$ having $\Delta z\,(u) \leq \delta z\,(u)$ can naturally be part of the heavy quark while those region of $u$ having $\Delta z\,(u) > \delta z\,(u)$ lag too far behind the core of the heavy quark to be part of it. Comparing Eq. (27) and Eq. (28), we see that the separation point, up to constant factor of order one, is given by Eq. (26). In Sec. V, we give a more complete discussion of this issue where, instead of relying on Eq. (28), we compare the four-dimensional energy-momentum tensor for a trailing string with that of the string corresponding to a heavy quark moving in the vacuum and reach the same conclusion as above.

Another argument for Eq. (26) as the natural separation point between what belongs to the heavy quark and what has been freed into the medium comes from the calculation of transverse momentum broadening from small fluctuation on the trailing string. In the differential equation governing the small fluctuations $u = u_s$ is a regular singular point which corresponds to a horizon of the metric on the worldsheet of the string[14, 15]. Waves on the worldsheet of the string going from large $u$ can disappear into the horizon but one requires that waves not come out of the horizon toward large values of $u$.[22, 23]

### F. Energy loss and $p_\perp$-broadening for finite matter and for dressed quarks

In dealing with hot matter of finite length in QCD, the energy loss problem is very different for dressed and bare quarks. For heavy ion collisions where high transverse momentum quarks are produced in the collision as bare quarks, without the accompanying gluon cloud

which characterize a dressed quark, the dressed quark energy loss problem is of little interest. Nevertheless we begin with this problem in our discussion of finite matter because it is relatively straightforward.

The situation is as follows: A high energy heavy quark prepared at early times impinges on a target of hot matter, say a cube, of length $L$. We wish to determine how much energy is lost and how much transverse momentum broadening occurs as the heavy quark passes through the medium. If the rapidity of the heavy quark is such that $\cosh\eta \ll (LT)^2$(For QCD the condition is $\cosh\eta \ll \sqrt{\hat{q}L^3}$), then the coherence time of all the gluons in the heavy quark is much less than $L$ and the matter is effectively of infinite extent. Thus, we consider here only the opposite case where $\cosh\eta \gg (LT)^2$. We also suppose $\frac{M}{LT^2} \gg 1$ so that a small fraction of the heavy quark's energy is lost as it passes through the matter.

When $\cosh\eta \gg (LT)^2$, the picture is essentially the same in QCD and for a SYM plasma. Gluons in the heavy quark wavefunction having $k_\perp < Q_s \sim LT^2$ will be freed in passing through the matter while those having $k_\perp \geq Q_s$ are not freed. Since the gluons which dominate the energy loss have $\omega \simeq k_\perp \cosh\eta$, the energy loss will be

$$-\frac{dE}{dt} \propto \begin{pmatrix} \alpha N_c \\ \sqrt{\lambda} \end{pmatrix} \frac{Q_s \cosh\eta}{L}. \tag{29}$$

with $\alpha N_c$ referring to QCD and $\sqrt{\lambda}$ to the SYM case. For SYM the result Eq. (29) agrees with Eq. (13), the infinite matter result, when Eq. (14) is used for $Q_s$. For QCD the result Eq. (29) differs from the case of infinite matter, and we remark that we have not subtracted the "factorization term"[24, 25]. We emphasize, however, that when written in terms of the saturation momentum the QCD and SYM results are of the same form once one makes the $\alpha N_c \leftrightarrow \sqrt{\lambda}$ identification in passing between theories.

For transverse momentum broadening the formulas are

$$\frac{dp_\perp^2}{dt} \propto \begin{pmatrix} 1 \\ \sqrt{\lambda} \end{pmatrix} \frac{Q_s^2}{L}. \tag{30}$$

where again the upper value is for QCD while the lower value is for a SYM plasma. The QCD result is the same as Eq. (17) for infinite matter while the SYM result takes the same form as infinite matter, as given in Eq. (22), but now one must use Eq. (14) for $Q_s^2$. The argumentation leading to Eq. (30) is identical to that giving Eq. (29) and, again, here in the SYM plasma transverse momentum broadening is radiation dominated.

### G. Energy loss and $p_\perp$-broadening for bare heavy quarks in finite extent matter

The quarks, whether light or heavy, that are produced in hard collisions in a relativistic heavy ion collision are initially bare, that is they have been produced without the gluon cloud that accompanies a quark which is part of a high energy hadron. While the energy loss of bare quarks has been widely discussed in the QCD literature[5, 6, 7, 8], it has so far not been treated in the literature on the SYM plasma. On the string theory side of the AdS/CFT correspondence, in order to create a heavy quark which is initially bare it is necessary to consider a heavy quark-antiquark pair initially with little or no separation and then rapidly accelerate the quark and the antiquark in opposite direction by, say, an electric field $E_f$ which is constant in space and time and points along the $z$-axis. After a rapid acceleration over a time $t_1$, the electric field can be turned off and one has a quark-antiquark pair with large relative energy in a situation similar to that occurring in a hard collision[1]. We have found an exact solution which satisfies the equations coming from the Nambu-Goto action for the quark-antiquark pair accelerating in the vacuum. This will be discussed in much more detail in Sec. IV and in Ref. [17], while here we give the solution and use it to outline how to get energy loss and $p_\perp$-broadening in a SYM plasma.

The solution, in the vacuum, described above is

$$z = \pm\sqrt{t^2 + \frac{c^2}{u_m^2} - \frac{1}{u^2}} \tag{31}$$

where the external electric field is

$$E_f = \frac{2\pi M^2}{\sqrt{\lambda}c} = \frac{\sqrt{\lambda}u_m^2}{2\pi c} \tag{32}$$

The quark is at $z > 0$ and antiquark is at $z < 0$. $M$ is the mass of the heavy quark as usual. The constant $c$ characterizes the strength of the electric field and $c \geq 1$ is required, as is clear from Eq. (31). We imagine $c$ to be large but we also suppose $\frac{u_m}{c}$ is very large compared to the temperature of the medium which we shall shortly introduce. At a time $t_1$, we shall turn off the electric field beyond which time Eq. (31) no longer applies. It is straightforward to see that, for $t < t_1$, the part of the string in the region $u_m/c < u < u_m$ moves with a rapidity $\eta\,(t)$ given by

$$\cosh\eta = \frac{1}{\sqrt{1-v^2}} \simeq \frac{u_m t}{c}. \tag{33}$$

We now suppose that

$$\frac{u_m t}{c} \gg 1, \tag{34}$$

however, we also imagine that $t_1$ is a small time compared to the inverse temperature of the medium. By choosing $u_m$ sufficiently large there is no difficulty in having $c \gg 1$, $u_m t_1/c \gg 1$, and $t_1 T \ll 1$ satisfied.

Before considering motion in a plasma, let us examine a little further the vacuum evolution of our heavy quark-antiquark system. When $c/u_m < t < t_1$, it is useful to consider the $z > 0$ part of the string as two parts. Part A consists of the $u_m/c \leq u \leq u_m$ region of the string, while part B is the $\frac{1}{\sqrt{t^2+c^2/u_m^2}} \leq u < u_m/c$ region of the string. Much as in the trailing string , part A is part of the heavy quark, while part B corresponds to radiated energy. Indeed with this interpretation we can calculate the power radiated at time $t$ as

$$P = \frac{d}{dt}\left[E_f t - \cosh\eta\left(t\right)\frac{\sqrt{\lambda}}{2\pi}\left(u_m - u_m/c\right)\right]. \tag{35}$$

The first term on the right hand side of Eq. (35) gives the rate at which the electric field puts energy into the system. Noting that the energy density of a string at rest is $\frac{dE}{du} = \frac{\sqrt{\lambda}}{2\pi}$, we see that the second term is the rate of growth of the energy in the "straight section" of the string in the region $u_m/c \leq u \leq u_m$. Eq. (35) gives

$$P_{\text{radiated}} = \frac{\sqrt{\lambda}}{2\pi}\frac{E_f^2}{M^2}. \tag{36}$$

We note that the answer for a classical electron accelerating in a constant electric field is[26]

$$P_{\text{electron}} = \frac{2e^2}{3}\frac{E_f^2}{M^2}, \tag{37}$$

parametrically the same as Eq. (36) with the replacement $e^2 \leftrightarrow \sqrt{\lambda}$.

Armed with our understanding that the part of the string $u < u_m/c$ is emitted radiation and no longer part of the heavy quark, we are now ready to insert the medium. We suppose that the acceleration during $0 < t < t_1$ takes place at one end of a hot SYM plasma whose $z$-extent is $L$. At $t = t_1$, we turn off the electric field and let our heavy quark pass through the length $L$ of the plasma. There are two separate cases to consider: (i), when $\cosh\eta\left(t_1\right) \equiv \cosh\eta \ll (LT)^2$ and (ii), when $\cosh\eta \gg (LT)^2$.

In case(i), the situation is just like that of a trailing string in an infinite medium. In vacuum the separation point between that part of the string corresponding to the heavy

quark and previously radiated energy is at $u = \frac{u_m}{c}\frac{t_1}{t} = \frac{\cosh\eta}{t}$ when $t > t_1$. At a time $t = \frac{\sqrt{\cosh\eta}}{T}$ this point crosses the separation point, $u = T\sqrt{\cosh\eta}$, of the trailing string at which point the heavy quark energy loss and transverse momentum broadening becomes that of the trailing string. This time at which the string becomes a trailing string is much less than $L$ so that the whole problem becomes that of an infinite medium trailing string problem. In case (i), $\cosh\eta$ is small enough that the system quickly adjust to the trailing string scenario.

In case(ii) the situation is much more subtle. To better appreciate the issues, let us first suppose the motion of the evolving string to take place in the vacuum. Then, when the electric field is turned off at $t = t_1$, the system will go toward a final configuration of a heavy quark moving in the vacuum at constant velocity along with some radiation. The radiation all goes toward $u = 0$ as explained in detail in Ref. [16]. For us the key issue is at what time the quanta making up the heavy quark are formed. We know that the separation point, which is at $u = u_m/c$ for $t < t_1$, moves down in $u$ as

$$(u)_{\text{separation}} = \frac{\cosh\eta}{t}, \tag{38}$$

when $t > t_1$, essentially according to free branching. However, once the electric field has been turned off this separation point need no longer be a perfect separation between what is part of the heavy quark and what is radiated energy. Although almost all the energy in the region $u < (u)_{\text{separation}}$ lags too far behind the core of the heavy quark to be part of it, at a distance

$$\Delta z\,(u,t) \sim \frac{1}{tu^2} \tag{39}$$

behind the core as is clear from Eq. (31), the components of the heavy quark having $k_\perp \sim u$ and $k_z \sim tu^2 \ll u\cosh\eta$ spatially overlap with the evolving string as is evident in the discussion in Sec. V and on Fig.1. If these components of the evolving string do contribute to the final heavy quark, they are formed at a time $t_c$ where

$$t_c \sim \frac{k_z}{u^2} \sim \frac{k_z}{k_\perp^2}, \tag{40}$$

which matches our expectation of the time at which such components are naturally formed. Of course the components of the heavy quark carrying most of the energy, having $k_z \simeq u\cosh\eta$, would still be formed much later at the time when the separation point, given in Eq. (38), reaches u.

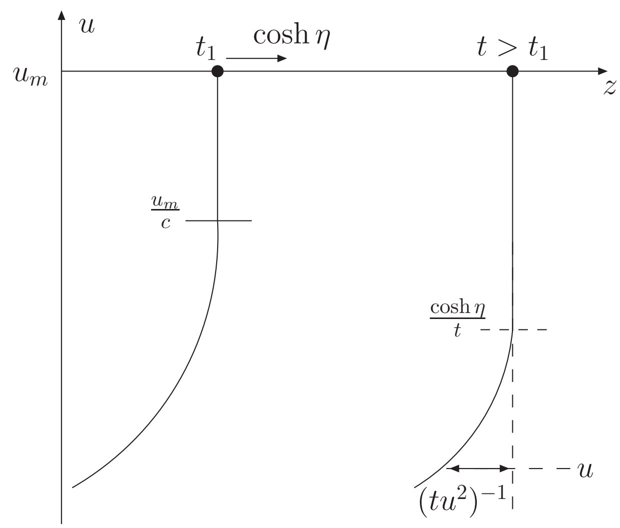


FIG. 1: The accelerating string in the vacuum. At the time $t_1$, when the electric field is turned off, the quark rapidity is $\cosh\eta\,(t_1) \equiv \cosh\eta$, and the part of the string with $u < u_m/c$ is emitted radiation. At a later time t, the heavy quark is rebuilding its gluon cloud, and for $u \ll \cosh\eta/t$, only soft components with $k_z < tu^2$ contribute to the heavy quark. When put in a medium of length L (such that $(LT)^2 \ll \cosh\eta$), radiated gluons must also have $u \leq Q_s$, therefore the hardest, dominant, radiations have $k_z = LQ_s^2$.

It is the identification of $\frac{k_z}{k_\perp^2} \simeq \frac{k_z}{u^2}$ with the time at which soft components, those where $\frac{k_z}{u\cosh\eta} \ll 1$, of the free heavy quark are formed which drives us toward identifying a (small) part of the evolving string having $u < (u)_{\text{separation}}$ as contributing to the heavy quark. We now make this conjecture, namely that after the electric field has been turned off at $t = t_1$, the parts of the heavy quark having $k_\perp \sim u$ and $k_z$ are formed at a time $t_c \sim k_z/u^2$ out of a small fraction, a fraction $\frac{k_z}{u\cosh\eta}$, of the energy of the evolving string located at $u$ at time on the order of $t_c$.

Now let us return to the string evolving in the SYM medium. After the electric field is turned off, the system goes toward radiation and a partially formed heavy quark passing through the medium. In our present situation, $\cosh\eta \gg (LT)^2$, the heavy quark is only fully formed after leaving the medium. Now medium induced energy loss is determined by which components of the heavy quark which are forming in the medium are inhibited from doing so by the medium. Clearly all momentum components $k_\perp \sim u < Q_s$ will be inhibited. Thus the energy loss will be equal to that part of the free heavy quark string having $u < Q_s$

and $\frac{k_z}{u^2} < L$, or

$$- \frac{dE}{dt} \sim \sqrt{\lambda} \frac{(k_z)_{max}}{L} \sim \sqrt{\lambda} Q_s^2, \tag{41}$$

exactly as in Eq. (16), but where now

$$Q_s \sim LT^2. \tag{42}$$

Similarly the transverse momentum broadening is given as

$$\frac{dp_\perp^2}{dt} \sim \sqrt{\lambda} \frac{dQ_s^2}{dL}, \tag{43}$$

as in Eq. (23) but now with Eq. (42) again determining $Q_s$.

The rather elaborate argument we have just given for energy loss, and which also give Eq. (42) for $p_\perp$-broadening is essentially identical to the understanding of energy loss in perturbative QCD in a finite length medium[5, 6, 7, 8, 13, 18]. So it should be no surprise that there the result is

$$- \frac{dE}{dt} = \frac{\alpha N_c}{4} \hat{q} L = \frac{\alpha N_c}{4} Q_s^2, \tag{44}$$

and

$$\frac{dp_\perp^2}{dt} = \hat{q} = \frac{dQ_s^2}{dL}, \tag{45}$$

when $\cosh \eta > \sqrt{\hat{q} L^3}$ and where the $Q_s^2$ in Eqs. (44) and (45) is given by $Q_s^2 = \hat{q} L$.

## III. LIENARD-WIECHERT CALCULATION OF GLUON DISTRIBUTION IN HEAVY QUARK.

It is known that the energy-momentum tensor of a heavy particle at rest in SYM theory has the same form as the one obtained from classical electrodynamics (except for a normalization factor)[10, 11]. This statement is easily generalized for a heavy particle moving at constant velocity since the energy-momentum tensor is related to the previous case by a Lorentz boost.

In this section we will show that, for a relativistic heavy particle, the classical result for the energy density in momentum space agrees with the quantum perturbative calculation to lowest order. For this purpose we will compute the energy density in momentum space by

Fourier transforming the Lienard-Wiechert potential of a moving charge at constant velocity and compare this result to the distribution of photons in the wavefunction of a fast moving charge.

For a particle moving at constant velocity along the z-axis, the potential is given by

$$A_0(x) = \frac{e}{4\pi\sqrt{(z-vt)^2 + (1-v^2)x_\perp^2}}, \tag{46}$$

$$A_z(x) = vA_0(x). \tag{47}$$

Taking a three-dimensional Fourier transform

$$A_0(\vec{k},t) = \frac{e}{(2\pi)^{3/2}} \frac{e^{-ivtk_z}}{k_\perp^2 + (1-v^2)k_z^2}. \tag{48}$$

The corresponding fields are

$$\vec{E}(\vec{k},t) = -\frac{ie^{-ivtk_z}}{(2\pi)^{3/2}} \frac{e(\vec{k}_\perp, (1-v^2)k_z)}{k_\perp^2 + (1-v^2)k_z^2} \tag{49}$$

$$\vec{B}(\vec{k},t) = i\vec{k} \times \vec{A}(\vec{k},t), \tag{50}$$

and the energy density

$$\frac{d\mathcal{E}}{d^2k_\perp dk_z} = \frac{e^2}{2(2\pi)^3} \frac{(1+v^2)k_\perp^2 + (1-v^2)^2 k_z^2}{(k_\perp^2 + (1-v^2)k_z^2)^2}. \tag{51}$$

For $v$ close to 1 we take

$$\frac{d\mathcal{E}}{d^2k_\perp dk_z} = \frac{e^2}{(2\pi)^3} \frac{k_\perp^2}{(k_\perp^2 + (1-v^2)k_z^2)^2}. \tag{52}$$

On the other hand, the wavefunction of a fast moving charge can be calculated perturbativelly to first order in the coupling

$$|\psi_p\rangle = |p\rangle + \sum_\lambda \int d^3k \ \psi_\lambda(\vec{k}) |p-k; k, \lambda\rangle \tag{53}$$

where

$$\psi_\lambda(\vec{k}) = \frac{1}{E_{\vec{p}-\vec{k}} + E_{\vec{k}} - E_{\vec{p}}} \langle p-k; k, \lambda | H_I | p\rangle \tag{54}$$

and $H_I$ is the interaction Hamiltonian. Assuming $p_z \gg m, k_z \gg k_\perp$ we find

$$\psi_\lambda(\vec{k}) = \frac{k_z}{k_\perp^2 + (1-v^2)k_z^2} \frac{e}{(2\pi)^{3/2}} \frac{1}{\sqrt{E_{\vec{p}}\, E_{\vec{p}-\vec{k}}\, |\vec{k}|}} p^\mu \epsilon_\mu^{(\lambda)}(\vec{k})^*. \tag{55}$$

In Coulomb gauge, the polarization vectors can be written as

$$\epsilon_\mu^{(\lambda)}(\vec{k}) = (0, \vec{\epsilon}_\perp^{(\lambda)}, -\frac{\vec{k}_\perp \cdot \vec{\epsilon}_\perp^{(\lambda)}}{k_z}), \tag{56}$$

so, when $v$ is close to 1, we find

$$\psi_\lambda(\vec{k}) = -\frac{e}{(2\pi)^{3/2}\sqrt{|\vec{k}|}} \frac{\vec{k}_\perp \cdot \vec{\epsilon}_\perp^{(\lambda)}(\vec{k})^*}{k_\perp^2 + (1-v^2)k_z^2}. \tag{57}$$

From this distribution we can extract the energy density in momentum space corresponding to the photons in the wavefunction.

$$\frac{d\mathcal{E}}{d^2k_\perp dk_z} = \sum_\lambda |\vec{k}||\psi_\lambda(\vec{k})|^2 \tag{58}$$

$$= \frac{e^2}{(2\pi)^3} \frac{k_\perp^2}{(k_\perp^2 + (1-v^2)k_z^2)^2}. \tag{59}$$

This is the same as (52), so we can conclude that the two pictures give the same result. Even though this calculation was done in the abelian case, the QCD calculation is analogous and gives the same result. Because the form of the energy momentum tensor in classical electrodynamics and strong coupling SYM theory is the same, the above calculation shows that when the strong classical fields of a high energy heavy quark in SYM theory are resolved into quanta, by requiring that the classical energy in a given wave number mode be given by the number of quanta in that mode times the mode frequency, the result is the same as in lowest order perturbation theory. Thus, despite the strong coupling of our SYM theory the distribution of numbers of quanta and energy in the various wave number modes is just the same as in lowest order perturbation theory up to a normalizing constant. Once we appreciate that the distribution of energy and momentum in modes is essentially the same as in lowest order perturbation it is no longer so surprising that the formulas for $dE/dt$ and $dp_\perp^2/dt$ have the same form as in perturbative QCD when expressed in terms of $Q_s$ since it is $Q_s$ which determines which modes are freed when the heavy quark passes through matter.

## IV. THE ACCELERATING STRING

### A. The accelerating string solution

We set up our accelerating string calculation as follows: a quark-antiquark pair is imbedded in a brane located at $u = u_m$, and a net electric field $E_f$ is imposed in the brane which

accelerates the quark and antiquark at a constant acceleration in their own proper frame (An additional small electric field $E_{f2}$ which balances the attractive force between the quark and antiquark is also understood.).

The metric of the resulting vacuum $Ad_5$ space can be written as

$$ds^2 = R^2\left[\frac{du^2}{u^2} - u^2dt^2 + u^2\left(dx^2+dy^2+dz^2\right)\right] \tag{60}$$

$$= \frac{R^2}{\bar{u}^2}\left(d\bar{u}^2 - dt^2 + dx^2 + dy^2 + dz^2\right), \tag{61}$$

where $R$ is the curvature radius of the $AdS_5$ space and $\bar{u} = \frac{1}{u}$. The dynamics of a classical string is characterized by the Nambo-Goto action,

$$S = -T_0 \int d\tau d\sigma \sqrt{-\det g_{ab}} \tag{62}$$

where $(\tau, \sigma)$ are the string world-sheet coordinates, and $-\det g_{ab} = -g$ is the determinant of the induced metric. $T_0$ is the string tension. We define $X^\mu(\tau,\sigma)$ as a map from the string world-sheet to the five dimensional space time, and introduce the following notation for derivatives: $\dot{X}^\mu = \partial_\tau X^\mu$ and $X'^\mu = \partial_\sigma X^\mu$. When one chooses a static gauge by setting $(\tau,\sigma) = (t,u)$, and defines $X^\mu = (t, u, x(t,u), 0, 0)$, it is straightforward to find that

$$-\det g_{ab} = \left(\dot{X}^\mu X'_\mu\right)^2 - \left(\dot{X}^\mu \dot{X}_\mu\right)\left(X'^\mu X'_\mu\right) \tag{63}$$

$$= R^4\left(1 - \dot{x}^2 + u^4 x'^2\right). \tag{64}$$

Therefore, the equation of motion of the classical string reads:

$$\frac{\partial}{\partial u}\left(\frac{u^4 x'}{\sqrt{-g}}\right) - \frac{\partial}{\partial t}\left(\frac{\dot{x}}{\sqrt{-g}}\right) = 0 \tag{65}$$

In general, this equation is a non-linear differential equation which involves two variables and two derivatives. Thus it is notoriously hard to solve directly when $x(t,u)$ is a non-trivial function of $(t,u)$. Fortunately, we have been able to find an exact solution which corresponds to the accelerating string. The solution reads,

$$x = \pm\sqrt{t^2 + b^2 - \frac{1}{u^2}} \tag{66}$$

where the $+$ part represents the right moving part of the string and the $-$ part yields the left moving part of the string, together with the smooth connection in the middle (see Fig. 2). The quark and antiquark are accelerating and moving away from each other. The constant

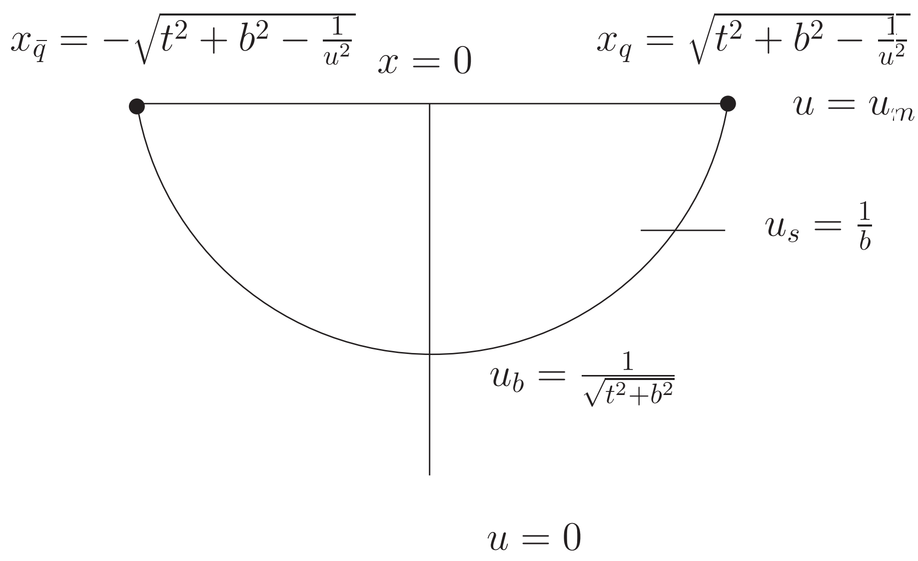


FIG. 2: Illustrating the accelerating string.

$b$ can be fixed by the boundary condition, and corresponds to $b = \frac{c}{u_m}$ from Sec. II, where $c$ is a dimensionless quantity. It is very easy to check that Eq. (66) satisfies the equation of motion by noting that $\sqrt{-g/R^4} = \frac{b}{\sqrt{t^2+b^2-\frac{1}{u^2}}}$.

Following Herzog et al [1] , one can compute the canonical momentum densities associated with the accelerating string,

$$\pi_\mu^0 = -T_0 \frac{\left(\dot{X}^\nu X'_\nu\right) X'_\mu - \left(X'^\nu X'_\nu\right) \dot{X}_\mu}{\sqrt{-g}}, \tag{67}$$

$$\pi_\mu^1 = -T_0 \frac{\left(\dot{X}^\nu X'_\nu\right) \dot{X}_\mu - \left(\dot{X}^\nu \dot{X}_\nu\right) X'_\mu}{\sqrt{-g}}. \tag{68}$$

The energy density is given by $\pi_t^0$,

$$\frac{dE}{du} = -\pi_t^0 = \frac{T_0 R^4}{\sqrt{-g}} \left(1 + u^4 x'^2\right). \tag{69}$$

Thus the total energy of the right half string at time $t$ is,

$$\int_{u_b}^{u_m} \frac{dE}{du} du = \frac{T_0 R^2 u_m}{b} \sqrt{t^2 + b^2 - \frac{1}{u_m^2}}. \tag{70}$$

Moreover, the energy flow is given by $\pi_t^1$,

$$\frac{dE}{dt} = \pi_t^1 = \frac{T_0 R^4}{\sqrt{-g}} u^4 x' \dot{x}. \tag{71}$$

Thus the net energy[1] being put into the right half string from 0 to $t$ is,

$$\int_0^t \frac{dE}{dt}dt\bigg|_{u=u_m} = \frac{T_0R^2u_m}{b}\left(\sqrt{t^2+b^2-\frac{1}{u_m^2}}-\sqrt{b^2-\frac{1}{u_m^2}}\right), \tag{72}$$

with the second term in the bracket being the initial energy deposited in the string. Also $b^2-\frac{1}{u_m^2}\geq 0$ is assumed for consistency. Therefore, from energy conservation, one can easily fix the constant $b$ by setting $E_f=\frac{T_0R^2u_m}{b}$, then,

$$b=\frac{M}{E_f}=\frac{\sqrt{\lambda}u_m}{2\pi E_f}, \tag{73}$$

where $M = T_0R^2u_m$ is the mass of the heavy quark and $T_0R^2 = \frac{\sqrt{\lambda}}{2\pi}$ according to the AdS/CFT correspondence. It is now very easy to see the physical interpretation of the constant $b$ as the reciprocal of the constant acceleration $a$, i.e., $a=\frac{E_f}{M}=\frac{1}{b}$.

In addition, although $\frac{\partial x}{\partial t}=\frac{t}{\sqrt{t^2+b^2-\frac{1}{u^2}}}$ exceeds 1 when $u$ becomes smaller than $1/b$, one can compute the speed at which energy travels by the following,

$$v=\frac{\partial x}{\partial t}+\frac{\partial x}{\partial u}\frac{du}{dt}=\frac{t}{t^2+b^2}\sqrt{t^2+b^2-\frac{1}{u^2}}, \tag{74}$$

and find that $v\leq 1$ at all times. In arriving at the above result, one needs to look at the hypersurface where energy is constant. Then, one can also obtain $\frac{du}{dt}=-\frac{\partial E}{\partial t}/\frac{\partial E}{\partial u}=-\frac{ut}{t^2+b^2}$. Finally, the Lorentz boost factor of the string reads,

$$\cosh\eta=\frac{1}{\sqrt{1-v^2}}=\frac{t^2+b^2}{\sqrt{(t^2+b^2)\,b^2+\frac{t^2}{u^2}}}, \tag{75}$$

and it reduces to $\frac{t}{b}=\frac{tu_m}{c}$ in the large $t$ and $u$ limits.

### B. Energy loss due to radiation

In a following paper[17], we will explicitly show that there is a scale $u_s=\frac{1}{b}$ separating the soft part(the lower part) of the string from the hard part of the string(the upper part).

[1] The total energy being put into the system should be the sum of the work done by $E_f$ and $E_{f2}$. $E_f$ is the field giving constant acceleration in the absense of Coulomb attraction between the heavy quark and antiquark. The field $E_{f2}$ is included to cancel that Coulomb force and is therefore time dependent. However only $E_f$ contributes to the non-Coulomb net energy increase and to the constant acceleration $E_f/M$.

The upper part, which moves together with the heavy quark, corresponds to the co-moving hard partons in the heavy quark wave function; The lower part ($u < u_s$) of the string, which lies far behind the heavy quark, is emitted radiation, and it is no longer part of the heavy quark.

Therefore, the radiated energy at time $t$ is

$$E_{\text{radiation}} = \int_{u_b}^{u_s} \frac{dE}{du} = \frac{\sqrt{\lambda}}{2\pi}\frac{t}{b^2}, \tag{76}$$

and the radiation power reads,

$$P = \frac{dE_{\text{radiation}}}{dt} = \frac{\sqrt{\lambda}}{2\pi}\frac{E_f^2}{M^2}, \tag{77}$$

in agreement with Eq. (35). By using the same picture, we can also estimate the $p_T$ and $p_L$ broadening due to radiation. At large time limit, one finds

$$\frac{dp_T^2}{dt} \propto \frac{\sqrt{\lambda}}{2\pi}\frac{u_s^2}{t} \sim \frac{\sqrt{\lambda}}{2\pi}\frac{1}{b^2 t}, \tag{78}$$

where $\sqrt{\lambda}$ basically counts the number of partons being emitted, $u_s^2$ is the typical momentum square of the emitted partons, and $t$ is just the time scale of the system. Similarly, one finds

$$\frac{dp_L^2}{dt} \propto \frac{\sqrt{\lambda}}{2\pi}\frac{\omega_s^2}{t} \sim \frac{\sqrt{\lambda}}{2\pi}\frac{t}{b^4}, \tag{79}$$

with $\omega_s \sim \frac{1}{\Delta x} \simeq u_s^2 t$ being the typical energy of the emitted partons and $\Delta x$ being the longitudinal separation between the quark and the string at $u = u_s$. Moreover, after identifying $u_s$ with $Q_s$, one discovers that the coherence time $t = \frac{\omega}{u_s^2}$ in this accelerating string scenario coincides with the one in QCD (see Eq. (3)).

The exact evaluation by employing random fluctuation analysis will be provided in Ref. [17], and it yields

$$\frac{dp_T^2(t)}{dt} = \frac{\sqrt{\lambda}}{\pi^2}\frac{1}{b^2\sqrt{t^2+b^2}} = \frac{\sqrt{\lambda}}{\pi^2}\frac{a^3}{\sqrt{a^2t^2+1}}, \tag{80}$$

$$\frac{dp_L^2(t)}{dt} = \frac{\sqrt{\lambda}}{2\pi^2}\frac{\sqrt{t^2+b^2}}{b^4} = \frac{\sqrt{\lambda}}{2\pi^2}a^3\sqrt{a^2t^2+1}. \tag{81}$$

Finally, we have checked that our solution Eq. (77) gives the same radiation as that of Mikhailov's general radiation formulas[27] when specialized to the case of constant acceleration.

### C. Accelerating string in non-zero temperature $AdS_5$ space

The metric of the resulting AdS black brane solution in 5 dimension with finite temperature $T$ can be written as

$$ds^2 = R^2 \left[ \frac{du^2}{u^2 \left(1 - \frac{u_h^4}{u^4}\right)} - u^2 \left(1 - \frac{u_h^4}{u^4}\right) dt^2 + u^2 \left(dx^2 + dy^2 + dz^2\right) \right], \qquad (82)$$

with $u_h = \pi T$. Unfortunately, we are unable to find an exact accelerating string solution in this case. Nevertheless, we can give some semi-quantitative discussion in the $a = 1/b = \frac{E_f}{M} \gg \pi T$ limit.

Suppose the quark-antiquark pair now is put in the non-zero temperature $AdS_5$ spacetime, and the electric field which accelerates the heavy quarks is imposed from $t = 0$ to $t = \infty$, we expect at time $t \simeq \frac{1}{b} \frac{1}{(\pi T)^2}$ that the string starts to lose a significant amount of energy into the plasma. In the string system, energy is flowing into the system from the top of the string, while it is being absorbed by the plasma from the bottom part of the string. As time increases, the acceleration slows down and energy loss to the plasma increases. As a result, the string trajectory starts to evolve towards the trailing string solution from the accelerating string solution. Ultimately, the energy loss per unit time will be the same as the energy gained per unit time from the electric field. Thus, the trailing string trajectory is fully formed and a final velocity is then reached. Therefore, one gets

$$\left(\frac{dE}{dt}\right)_{\text{loss}} + \left(\frac{dE}{dt}\right)_{\text{gain}} = 0. \qquad (83)$$

Using Eq. (13) for energy loss per unit time and $E_f$ for energy gain per unit time, one gets the final $\cosh \eta_f$:

$$\cosh \eta_f \simeq \frac{2\pi E_f}{\sqrt{\lambda}\,(\pi T)^2} \qquad (84)$$

In arriving above formula, we have assumed that $\cosh \eta_f \gg 1$. $\cosh \eta_f$ is independent of $u_m$ as a result of energy balance between the external electric field and dissipative plasma. In the end, we can estimate the typical time which the string needs to reach the trailing string solution from Eq. (84) and Eq. (33), and obtain,

$$t_f \sim \frac{2\pi M}{\sqrt{\lambda}\,(\pi T)^2}. \qquad (85)$$

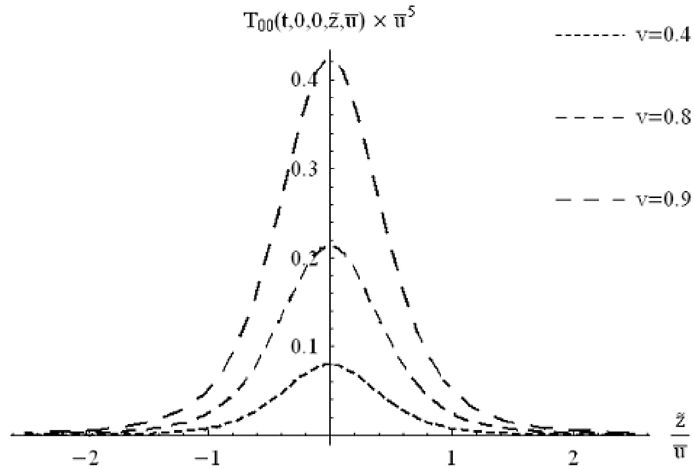


FIG. 3: $T_{00}(t,0,0,\tilde{z},\bar{u})$ of the straight string with velocity $v = 0.4, 0.8,$ and $0.9$ as a function of $\tilde{z}/\bar{u}$. $T_{00}$ peaks at $\tilde{z} = 0$ and has a width $\Delta\tilde{z} \simeq \bar{u}$, which implies that a portion of straight string around $\bar{u}$ can only be projected predominantly onto the energy density of the quark with $|\tilde{z}| \leq \bar{u}$.

## V. ENERGY MOMENTUM TENSOR IN 4-DIMENSION FROM TRAILING AND ACCELERATING STRINGS

In this section, we will give a quantitative confirmation of our picture by investigating $T_{\mu\nu}(t,\vec{x})$, the four-dimensional energy-momentum tensor of the "gluonic" field around a heavy quark. In the large $N_c$ limit, $T_{\mu\nu}(t,\vec{x})$ can be obtained from $h_{MN}$, the small metric perturbations to the $AdS_5$ metric $G^{(0)}_{MN}$ in the presence of the classical string in $AdS_5$ space, corresponding to either the trailing string or the accelerating string. We are only dealing with the energy density, i.e., the $(0,0)-$component of $T_{\mu\nu}$. It can be obtained from $h_{00}$, one of the scalar modes in the Fourier decomposition of metric perturbations in position space $R^3$.

### A. Energy density for the straight string with constant velocity $v$ in the vacuum

As argued in Sec. II, our criteria to distinguish between the quark part and the radiated energy part of a (trailing or accelerating) string is to compare it with a straight string moving with the same velocity $v$ in the vacuum and see which part is like that of a straight string and which part is lagging far behind it. To see how a portion of a straight string around $\bar{u} \equiv \frac{1}{u}$ with velocity $v$ is related to its energy density in position space, we define the energy

density in $(\vec{x},\bar{u})$ space for the straight string as [2]

$$T_{00}^{ss}(t,\vec{x},\bar{u}) \equiv -\frac{R^3}{6\kappa_5^2}\int \frac{d^4k}{(2\pi)^4}k^2\frac{S(k)}{Q^2}K_2(Q\bar{u})e^{ikx}, \tag{86}$$

where $Q=\sqrt{k^2-\omega^2}$ and the source term[10]

$$S(k) = -\frac{\kappa_5^2\sqrt{\lambda}}{R^3}\frac{[k_\perp^2(2+v^2)+2(1-v^2)k_3^2]Q^2}{k^2\sqrt{1-v^2}}\delta(\omega-vk_3). \tag{87}$$

Inserting $S(k)$ into (86), we get

$$T_{00}^{ss}(t,\vec{x},\bar{u}) = \frac{\sqrt{\lambda}}{8\pi^2(1-v^2)}\left[\frac{5\bar{u}^2}{(\bar{u}^2+\tilde{x}^2)^{7/2}}+\frac{v^2}{6}\frac{10\bar{u}^4-2\tilde{x}^4+6\tilde{x}^2\tilde{z}^2-7\bar{u}^2(\tilde{x}^2-3\tilde{z}^2)}{\bar{u}^2(\bar{u}^2+\tilde{x}^2)^{7/2}}\right], \tag{88}$$

where $\tilde{z}\equiv\frac{z-vt}{\sqrt{1-v^2}}$ and $\tilde{x}=\sqrt{x^2+y^2+\tilde{z}^2}$.

Notice that, as shown in Fig.3, a portion of straight string around $\bar{u}$ can only be projected predominantly onto the energy density of the quark with $|\tilde{z}|\leq\bar{u}$, that is $|z(\bar{u})-vt|\leq\frac{\bar{u}}{\cosh\eta}$. This is just the infrared-ultraviolet correspondence. Also, after adding missing terms [10] and integrating out $\bar{u}$, we have

$$T_{00}^{ss}(t,\vec{x}) = \frac{\sqrt{\lambda}}{12\pi^2(1-v^2)}\frac{(1+v^2)\tilde{x}^2-2v^2\tilde{z}^2}{\tilde{x}^6}. \tag{89}$$

### B. Saturation momentum from the four-dimensional energy-momentum tensor of the trailing string

For the trailing string, the equation of motion of the scalar mode is much more complicated[10, 11], and we will only keep terms up to $\mathcal{O}(T^2)$. In this approximation, we can evaluate $T_{00}^{ts}(t,\vec{x},\bar{u})$ as

---

[2] Here, we use the equation of motion for the scalar field given in Ref. [10]. Note that our definition of $T_{00}(t,\vec{x},\bar{u})$ is not quite such that $\int d\bar{u}T_{00}(t,\vec{x},\bar{u}) = T_{00}(t,\vec{x})$, but its purpose is rather to explain our picture of Section II. In the accelerating string calculation, $T_{00}^{as}(t,\vec{x},\bar{u})$ will be defined from the master equation of Ref. [11].

$$
\begin{aligned}
T^{ts}_{00}(t,\vec{x},\bar{u}) \equiv & -\frac{R^3}{6\kappa_5^2}\int\frac{d^4k}{(2\pi)^4}k^2\frac{S(k)}{Q^2}K_2(Q\bar{u})\exp\{-i\omega(t-\frac{1}{3}(\pi T)^2\bar{u}^3)+ik_\perp\cdot x_\perp+ik_3 z\}\\
=&\frac{\sqrt{\lambda}}{12\pi\sqrt{1-v^2}}\int\frac{d^3k}{(2\pi)^3}[k_\perp^2(2+v^2)+2(1-v^2)k_3^2]K_2(Q\bar{u})\exp\{ik_\perp\cdot x_\perp+ik_3\bar{z}\}\\
=&\frac{\sqrt{\lambda}}{8\pi^2(1-v^2)}\left[\frac{5\bar{u}^2}{(\bar{u}^2+\tilde{x}^2)^{7/2}}+\frac{v^2}{6}\frac{10\bar{u}^4-2\tilde{x}^4+6\tilde{x}^2\tilde{z}^2-7\bar{u}^2(\tilde{x}^2-3\tilde{z}^2)}{\bar{u}^2(\bar{u}^2+\tilde{x}^2)^{7/2}}\right].
\end{aligned}
\tag{90}
$$

where $\bar{z}\equiv z-vt+\frac{v}{3}(\pi T)^2\bar{u}^3$, $\tilde{z}\equiv\frac{\bar{z}}{\sqrt{1-v^2}}$ and $\tilde{x}=\sqrt{x^2+y^2+\tilde{z}^2}$. Comparing the expressions for $T_{00}(t,\vec{x},\bar{u})$ of the straight string and trailing string, the only difference is in the definition of $\tilde{z}$ in the two cases. If $\frac{v}{3}(\pi T)^2\bar{u}^3\ll\frac{\bar{u}}{\cosh\eta}$, there is essentially no difference between the straight and trailing strings in the region $|z-vt|\leq\frac{\bar{u}}{\cosh\eta}$ where almost all of the energy of the straight string is located. If $\frac{v}{3}(\pi T)^2\bar{u}^3\gg\frac{\bar{u}}{\cosh\eta}$, the contribution to the energy of the trailing string is much more spread out in $z-vt$ than for the straight string and we conclude that this part of the trailing string is not actually a part of the heavy quark but represents energy radiated in the medium. We note that the transition point between parts of the trailing string that belong the heavy quark and those that are waves in the plasma is at $(\pi T)^2\bar{u}^3\sim\frac{\bar{u}}{\cosh\eta}$ or $u\sim\pi T\sqrt{\cosh\eta}$ which is the natural separation point discussed in Sec. II.

This separation point can also be obtained when looking at the $\bar{u}-$integrated $T_{00}$, in the rest frame of the heavy quark: Ref. [28] gives all the components of $T_{\mu\nu}$ (up to $\mathcal{O}(T^2)$) in the plasma rest frame, and boosting to the quark rest frame gives

$$
\bar{T}^{00}_{ts}\frac{\sqrt{\lambda}}{12\pi^2x^4}-\frac{5\sqrt{\lambda}}{72}T^2\cosh\eta\frac{vz}{x^3}+\mathcal{O}(T^4),
\tag{91}
$$

where the contribution of the plasma alone has been subtracted. Equating the two terms, one sees that the quark field is unchanged when $(\pi T)^2\cosh\eta vz|x|\lesssim 1$. On the longitudinal axis ($x=y=0$), this means $Q_s z\lesssim 1$ with $Q_s=\pi T\sqrt{\cosh\eta}$. Or more generally, so long as $|\vec{x}|\leq 1/Q_s$ the energy density agrees well with that of a heavy quark in the vacuum while when $|\vec{x}|\geq 1/Q_s$ there are strong medium modifications.

### C. Energy density for an accelerating string in the vacuum

For the accelerating string $x^2 = t^2+b^2-\frac{1}{u^2}$, the classical stress energy tensor in momentum space is (with $M, N = (t, x, y, z, u/R^2)$)

$$t_{MN}(t,\vec{k},u) = \frac{\kappa_5^2}{\pi\alpha'}\Theta\left(u-\frac{1}{\sqrt{t^2-x^2+b^2}}\right)\frac{1}{Ru}$$

$$\times\begin{pmatrix} \frac{b^2+t^2}{bx}\cos(k_1x) & i\frac{t}{b}\sin(k_1x) & 0 & 0 & \frac{1}{(Ru)^3}\frac{Rt}{bx}\cos(k_1x) \\ i\frac{t}{b}\sin(k_1x) & \frac{1}{u^2}\frac{t^2u^2-1}{bx}\cos(k_1x) & 0 & 0 & i\frac{1}{(Ru)^3}\frac{R}{b}\sin(k_1x) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{(Ru)^3}\frac{Rt}{bx}\cos(k_1x) & i\frac{1}{(Ru)^3}\frac{R}{b}\sin(k_1x) & 0 & 0 & \frac{1}{R^4u^6}\frac{1-b^2u^2}{bx}\cos(k_1x) \end{pmatrix}, \tag{92}$$

where $x = \sqrt{t^2+b^2-\frac{1}{u^2}}$. To simplify our equations below, we will use $\bar{u} = 1/u$ in the following calculation. In the zero temperature case, the master equation for the scalar modes of the metric perturbation due to the presence of classical string solutions is [11]

$$\varphi_s''(\bar{u}) + \frac{1}{\bar{u}}\varphi_s'(\bar{u}) - (k^2-\omega^2)\varphi_s(\bar{u}) = -J_s(\bar{u}), \tag{93}$$

where the scalar master field $\varphi_s(\bar{u}) = \frac{P_s}{\bar{u}} + Q_s^{tot} + ...$, which gives the energy density as $T_{00}(t,\vec{x}) = \int\frac{d^4k}{(2\pi)^4}e^{ikx}Q_s^{tot}$ and the source term

$$J_s(\bar{u}) = \int dt e^{i\omega t}S(\bar{u},t), \tag{94}$$

is a linear combination of $t_{MN}$[11], and $S(\bar{u},t)$ for the accelerating string is as follows,

$$\begin{aligned} S(\bar{u},t) \equiv & -\frac{\sqrt{\lambda}}{6bk^2\bar{u}^5\omega x^3} \\ & \Big\{(k_1\sin(k_1x)x(2k^2\bar{u}^2(it(-3+k^2\bar{u}^2)+3\bar{u}^2\omega)x^2+6(k^2-3k_1^2)R^4(t^2-\bar{u}^2)\omega x'^2+ \\ & R^2x(\bar{u}(2ik^4t\bar{u}^2-3(k^2-3k_1^2)(3t^2+\bar{u}^2)\omega)x'-3(k^2-3k_1^2)R^2(t^2-\bar{u}^2)\omega x''))+ \\ & \cos(k_1x)(-6ik^2k_1^2R^2t\bar{u}x^3x'+6(k^2-3k_1^2)R^4(t^2-\bar{u}^2)\omega x'^2+ \\ & x^2(\bar{u}^2(2k^4\bar{u}^2(2it-(b^2+2t^2-3\bar{u}^2)\omega)-9k_1^2(t^2-\bar{u}^2)\omega(1+\bar{u}^2\omega^2)+ \\ & 3k^2(t^2-\bar{u}^2)\omega(1+\bar{u}^2(4k_1^2+\omega^2)))-3k_1^2(k^2-3k_1^2)R^4(t^2-\bar{u}^2)\omega x'^2)+ \\ & R^2x(\bar{u}(2ik^4t\bar{u}^2-3(k^2-3k_1^2)(3t^2+\bar{u}^2)\omega)x'-3(k^2-3k_1^2)R^2(t^2-\bar{u}^2)\omega x'')))\Big\} \\ = & S_q(\bar{u},t)e^{-ik_1x_{as}(\bar{u},t)} + S_{\bar{q}}(\bar{u},t)e^{ik_1x_{as}(\bar{u},t)}, \end{aligned} \tag{95}$$

with $x = x_{as}(\bar{u}, t) = \sqrt{t^2 + b^2 - \bar{u}^2}$. From (93), it is easy to show that

$$Q_s^{tot} = \lim_{\epsilon \to 0} \{ \int_{\epsilon}^{\infty} d\bar{u}\bar{u} K_0(Q\bar{u}) J_s(\bar{u}) - \text{ poles at } 0 \}. \tag{96}$$

To compare an accelerating string with a straight string, we write

$$\begin{aligned} T_{00}^{as}(t, \vec{x}, u) &= \int \frac{d^4k d\tilde{t}}{(2\pi)^4} \bar{u} K_0(Q\bar{u}) S(\bar{u}, \tilde{t}) e^{ikx} \\ &= \int \frac{d^4k d\tilde{t}}{(2\pi)^4} \bar{u} K_0(Q\bar{u}) \\ &\quad \times \left\{ S_q(\bar{u}, \tilde{t}) \exp[-i\omega(t - \tilde{t}) + ik_1(x - x_{as}(\bar{u}, \tilde{t})) + ik_\perp \cdot x_\perp] \right. \\ &\quad \left. + S_{\bar{q}}(\bar{u}, \tilde{t}) \exp[-i\omega(t - \tilde{t}) + ik_1(x + x_{as}(\bar{u}, \tilde{t})) + ik_\perp \cdot x_\perp] \right\}, \end{aligned} \tag{97}$$

or, in momentum space,

$$T_{00}^{as}(\omega, \vec{k}, u) = \bar{u} K_0(Q\bar{u}) J_s(\bar{u}). \tag{98}$$

The Bessel $K$ functions and Fourier transformations between position space and momentum space ensure the infrared-ultraviolet correspondence. We will evaluate $T_{00}^{as}(t, \vec{x}, u)$ or $T_{00}^{as}(\omega, \vec{k}, u)$ in the approximation $t >> b, \bar{u}$ and only keep terms of order $\mathcal{O}(\frac{t}{b})$. In (97), we integrate over the region $t - \mathcal{T} < \tilde{t} < t + \mathcal{T}$ where we choose a $\mathcal{T}$ which is much smaller than t but large enough to ensure we can use the $\delta$ function as a good approximation to the resulting $\tilde{t}$ integration. In our calculation we consider the contribution to $T_{00}(t, \vec{x}, \bar{u})$ of an element of matter on the string, located at $\bar{u}$ at time $t$, and moving along the string at velocity $v(\bar{u}, t) = \frac{x_{as}(\bar{u}, t)}{t^2 + b^2} \simeq 1 - \frac{\bar{u}^2}{2t^2}$ as given in (74) of Sec. V. In case (i) below where $\bar{u}$ is small it will not be necessary to distinguish between $\dot{x}_{as}(\bar{u}, t) = \frac{\partial}{\partial t} x_{as}(\bar{u}, t)$ and $v(\bar{u}, t)$ since $\frac{\partial}{\partial \bar{u}} x_{as}(\bar{u}, t)$ is so small that the difference of these two velocities is unimportant. In case (ii) below where $\bar{u}$ is relatively large this distinction is important, and the natural organization of the calculation is to evaluate the contribution to $T_{00}$ of a fixed element of matter, following its motion on the string, over a period of time $2\mathcal{T}$. Of course this procedure is effective only because the motion of matter on the string between time $t - \mathcal{T}$ and $t + \mathcal{T}$ is small, $\bar{u}(t + \mathcal{T}) - \bar{u}(t - \mathcal{T}) \simeq \frac{\partial \bar{u}}{\partial t} 2\mathcal{T} \simeq \bar{u}\frac{2\mathcal{T}}{t} \ll \bar{u}$ when $\frac{2\mathcal{T}}{t} \ll 1$. In this large $t$ approximation, we

have

$$\begin{aligned}
T^{as}_{00}(t,\vec{x},u) &= \int \frac{d^4k}{(2\pi)^4}\bar{u}\int_{t-\mathcal{T}}^{t+\mathcal{T}} d\tilde{t} K_0(Q\bar{u}) \\
&\quad \times \{S_q(\bar{u},\tilde{t})\exp[-i\omega(t-\tilde{t}) + ik_1(x - x_{as}(\bar{u},\tilde{t})) + ik_\perp\cdot x_\perp] \\
&\quad + S_{\bar{q}}(\bar{u},\tilde{t})\exp[-i\omega(t-\tilde{t}) + ik_1(x + x_{as}(\bar{u},\tilde{t})) + ik_\perp\cdot x_\perp]\} \\
&\simeq \int \frac{d^4k}{(2\pi)^4}\bar{u}\int_{-\mathcal{T}}^{\mathcal{T}} d\bar{t} K_0(Q\bar{u}) \\
&\quad \times \{S_q(\bar{u},t)\exp[i\omega\bar{t} + ik_1(x - x_{as}(\bar{u},t) - v(\bar{u},t)\bar{t}) + ik_\perp\cdot x_\perp] \\
&\quad + S_{\bar{q}}(\bar{u},t)\exp[i\omega\bar{t} + ik_1(x + x_{as}(\bar{u},t) + v(\bar{u},t)\bar{t}) + ik_\perp\cdot x_\perp]\} \\
&\simeq \int \frac{d^4k}{(2\pi)^3}\bar{u}K_0(Q\bar{u}) \\
&\quad \times \{\delta(\omega - k_1 v(\bar{u},t))S_q(\bar{u},t)\exp[ik_1(x - x_{as}(\bar{u},t)) + ik_\perp\cdot x_\perp] \\
&\quad + \delta(\omega + k_1 v(\bar{u},t))S_{\bar{q}}(\bar{u},t)\exp[ik_1(x + x_{as}(\bar{u},t)) + ik_\perp\cdot x_\perp]\}
\end{aligned} \tag{99}$$

Therefore, in the end, we arrive at

$$T^{as}_{00}(t,\vec{x},u) \simeq \int \frac{d^4k}{(2\pi)^4}\bar{u}K_0(Q\bar{u})J_s(\bar{u}), \tag{100}$$

We will compare the following two cases:

(i) $t \gg b \gg \bar{u}$,

In our picture, the two parts of the accelerating string are the parts of a quark and an anti-quark respectively. We can show that in the large $t$ limit, the energy density related to these two parts of the string is simply a sum of that of a quark and an anti-quark. In this case, $v(\bar{u},t) \simeq 1$ and the source term $J_s(\bar{u})$ is

$$\begin{aligned}
J_s(\bar{u}) \simeq & \frac{\sqrt{\lambda}t}{b}\Bigg\{\delta(\omega - k_1)\left[-\frac{2k_1^2}{\bar{u}} - \frac{\omega^2}{2\bar{u}} + \frac{2k^2}{3\bar{u}} + \frac{k_1k^2}{3\bar{u}\omega} + \frac{3k_1^2\omega^2}{2k^2\bar{u}} - \frac{1}{2\bar{u}^3} + \frac{3k_1^2}{2k^2\bar{u}^3} - \frac{k_1}{\bar{u}^3\omega}\right] \\
& + \delta(\omega + k_1)\left[-\frac{2k_1^2}{\bar{u}} - \frac{\omega^2}{2\bar{u}} + \frac{2k^2}{3\bar{u}} - \frac{k_1k^2}{3\bar{u}\omega} + \frac{3k_1^2\omega^2}{2k^2\bar{u}} - \frac{1}{2\bar{u}^3} + \frac{3k_1^2}{2k^2\bar{u}^3} + \frac{k_1}{\bar{u}^3\omega}\right]\Bigg\},
\end{aligned} \tag{101}$$

where $\frac{t}{b} \simeq \cosh\eta_0$ in the large t limit. Compared with the source term of a straight string

with velocity $v$[11],

$$J_s^{ss}(\bar{u}) = \frac{\sqrt{\lambda}\delta(\omega - vk_1)}{\sqrt{1-v^2}}\left[-\frac{2k_1^2v^2}{\bar{u}} - \frac{v^2\omega^2}{2\bar{u}} + \frac{(1+v^2)k^2}{3\bar{u}} + \frac{k^2k_1v}{3\bar{u}\omega} + \frac{3k_1^2v^2\omega^2}{2k^2\bar{u}} - \frac{v^2}{2\bar{u}^3} + \frac{3k_1^2v^2}{2k^2\bar{u}^3} - \frac{k_1v}{\bar{u}^3\omega}\right]$$
$$\xrightarrow{v\to 1} \frac{\sqrt{\lambda}\delta(\omega - k_1)}{\sqrt{1-v^2}}\left[-\frac{2k_1^2}{\bar{u}} - \frac{\omega^2}{2\bar{u}} + \frac{2k^2}{3\bar{u}} + \frac{k^2k_1}{3\bar{u}\omega} + \frac{3k_1^2\omega^2}{2k^2\bar{u}} - \frac{1}{2\bar{u}^3} + \frac{3k_1^2}{2k^2\bar{u}^3} - \frac{k_1}{\bar{u}^3\omega}\right], \tag{102}$$

so that, $J_s(\bar{u})$ is simply a sum of that of a quark and an anti-quark flying apart with constant velocity $v$ close to 1. Thus in the region $\bar{u} \ll b$ and for $t \gg b$ the energy density of an accelerating string is indistinguishable from that of a string moving at constant velocity with $\frac{t}{b} = \cosh\eta_0$.

(ii)$t \gg \bar{u} \gg b$,

In our picture, this part of string corresponds to radiated energy and we will see that it contributes predominantly to the energy density far behind the quark. Recalling that in this case $v(\bar{u},t) = \frac{x_{as}(\bar{u},t)}{t^2+b^2} \simeq 1 - \frac{\bar{u}^2}{2t^2}$, and $\cosh\eta \simeq \frac{t}{\bar{u}}$, let us focus only on the half string associated with a quark, where the energy density at $(\vec{x},\bar{u})$ is

$$\begin{aligned} T_{00}^{as}(t,x,0_\perp,u) =& \int \frac{d^4k}{(2\pi)^4}\exp\{ik_1[x - x_{as}(\bar{u},t)]\}\bar{u}K_0(Q\bar{u})J_s(\bar{u})\} \\ \simeq& \frac{\sqrt{\lambda}t}{b}\int \frac{d^4k}{(2\pi)^4}\bar{u}K_0(Q\bar{u})\exp\{ik_1[x - x_{as}(\bar{u},t)]\} \\ &\times\left\{\delta(\omega - k_1(1-\frac{\bar{u}^2}{2t^2}))\left[-\frac{2k_1^2}{\bar{u}} - \frac{\omega^2}{2\bar{u}} + \frac{2k^2}{3\bar{u}} + \frac{k_1k^2}{3\bar{u}\omega}\right.\right. \\ &\left.\left.+\frac{3k_1^2\omega^2}{2k^2\bar{u}} - \frac{1}{2\bar{u}^3} + \frac{3k_1^2}{2k^2\bar{u}^3} - \frac{k_1}{\bar{u}^3\omega} + \mathcal{O}(\frac{u}{t})\right]\right\} \\ \simeq& \frac{\sqrt{\lambda}t}{b}\int \frac{dk_1dk_\perp k_\perp}{(2\pi)^3}K_0(\tilde{k}\bar{u})\exp\{ik_1[x - x_{as}(\bar{u},t)]\} \\ &\times\left[-\frac{5k_1^2}{2} + k^2 + \frac{3k_1^4}{2k^2} - \frac{3}{2\bar{u}^2} + \frac{3k_1^2}{2k^2\bar{u}^2} + \mathcal{O}(\frac{u}{t})\right], \end{aligned} \tag{103}$$

where $\tilde{k} \equiv \sqrt{\frac{\bar{u}^2}{t^2}k_1^2 + k_\perp^2}$. The exponential decrease of the Bessel $K_0$ function at large values of its argument requires $\frac{\bar{u}}{t}k_1, k_\perp \sim 1/\bar{u}$, which means $\frac{k_1^2}{k^2} \simeq 1$ and $\frac{k_\perp^2}{k_1^2} \ll 1$, so in this approximation, we have

$$\begin{aligned} T_{00}^{as}(t,x,0_\perp,u) \simeq& \frac{\sqrt{\lambda}t}{b}\int \frac{dk_1dk_\perp k_\perp^3}{(2\pi)^3}K_0\left(\tilde{k}\bar{u}\right)\exp\left\{ik_1[x - x_{as}(\bar{u},t)]\right\} \\ =& \frac{3\sqrt{\lambda}}{4\pi^2}\frac{t}{b}\frac{t}{\bar{u}}\frac{1}{(\bar{u}^2 + \frac{t^2}{u^2}\bar{x}^2)^{5/2}}, \end{aligned} \tag{104}$$

where $\bar{x} \equiv x - x_{as}(\bar{u}, t)$. The dominant term of $T_{00}^{as}(t, x, 0_\perp, \bar{u})$ of order $\mathcal{O}(\frac{t}{b})$ shows that this part of the accelerating string indeed contributes predominantly around $x_{as}(\bar{u}, t)$ with $\Delta(x - x_{as}(\bar{u}, t)) \simeq \frac{\bar{u}^2}{t} = \frac{\bar{u}}{\cosh\eta}$ while $x_{as}(0, t) - x_{as}(\bar{u}, t) \simeq \frac{\bar{u}^2}{2t}$. Although there is some contribution which could be associated with the energy density of the quark, it is quite small relative to the part of string with $\bar{u} \ll b$, since it is suppressed by a factor $\cosh\eta/\cosh\eta_0 = \frac{b}{\bar{u}} \ll 1$ due to their different Lorentz contractions of the local part of the string at $\bar{u}$ and that of the overall motion of the heavy quark. On the other hand, the quark part of the string with $\bar{u} \ll b$ barely contributes to this part of energy of the "gluonic" field since its contribution is more localized around the position of the quark due to much larger Lorentz contraction $\cosh\eta_0 = \frac{t}{b}$.

### D. Seeing the separation point on the trailing string from causality

There is a simple argument, although not directly related to the energy-momentum tensor, which shows that the point $u = \pi T\sqrt{\cosh\eta} = Q_s$ separates the trailing string into a part which is causally connected from a part which is not causally connected to the core of the heavy quark. In $AdS_5$ space, we can boost the plasma rest frame into the quark and the trailing string rest frame by the following generalized Lorentz transformation,

$$\hat{x} = \cosh\eta\,[x - v(t + F(u))]\,, \tag{105}$$

$$\hat{t} = \cosh\eta\left[t - vx + F(u) - \cosh^{-\frac{3}{2}}\eta F\left(\frac{u}{\sqrt{\cosh\eta}}\right)\right], \tag{106}$$

$$\hat{y} = y, \hat{z} = z, \hat{u} = u, \tag{107}$$

where $F(u) = \frac{1}{2u_h}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{u}{u_h}\right) - \coth^{-1}\left(\frac{u}{u_h}\right)\right]$ and we take $z_0 = 0$ in (24). In this frame, the trailing string appears as a straight string sitting at $\hat{x} = 0$ at all time $\hat{t}$, and the $AdS_5$ metric is

$$\begin{aligned} ds^2 =& R^2u^2\left[-\hat{f}(u)d\hat{t}^2 + 2v\cosh^2\eta\frac{u_h^4}{u^4}d\hat{x}d\hat{t} + (1 + v^2\cosh^2\eta\frac{u_h^4}{u^4})d\hat{x}^2 + dy^2 + dz^2\right] \\ &+ \frac{2R^2v\cosh\eta u_h^2 d\hat{x}du}{u^2\hat{f}(u)} + \frac{du^2R^2}{u^2\hat{f}(u)}, \end{aligned} \tag{108}$$

where $\hat{f}(u) \equiv 1 - \frac{u_h^4\cosh^2\eta}{u^4}$. From this metric, it is easy to see that it takes a beam of light emitted at $u = u_h\sqrt{\cosh\eta} = \pi T\sqrt{\cosh\eta}$, the separation point that we have discussed earlier,

an infinite period of time $\Delta\hat{t} = \Delta t/\cosh\eta$ to reach the quark, that is, the quark can never "see" the part of the string with $u \leq \pi T\sqrt{\cosh\eta}$.

---


[1] C. P. Herzog, A. Karch, P. Kovtun, C. Kozcaz and L. G. Yaffe, JHEP **0607**, 013 (2006).

[2] S. S. Gubser, Phys. Rev. D **74**, 126005 (2006) [arXiv:hep-th/0605182].

[3] H. Liu, K. Rajagopal and U. A. Wiedemann, Phys. Rev. Lett. **97**, 182301 (2006) [arXiv:hep-ph/0605178].

[4] Y. L. Dokshitzer and D. E. Kharzeev, Phys. Lett. B **519**, 199 (2001) [arXiv:hep-ph/0106202].

[5] R. Baier, Y. L. Dokshitzer, A. H. Mueller and D. Schiff, Nucl. Phys. B **531**, 403 (1998) [arXiv:hep-ph/9804212].

[6] B. G. Zakharov, JETP Lett. **63**, 952 (1996) [arXiv:hep-ph/9607440]; JETP Lett. **65**, 615 (1997) [arXiv:hep-ph/9704255].

[7] M. Gyulassy, P. Levai and I. Vitev, Phys. Rev. Lett. **85**, 5535 (2000) [arXiv:nucl-th/0005032];Nucl. Phys. B **594**, 371 (2001) [arXiv:nucl-th/0006010].

[8] C. A. Salgado and U. A. Wiedemann, Phys. Rev. D **68**, 014008 (2003) [arXiv:hep-ph/0302184].

[9] Y. Hatta, E. Iancu and A. H. Mueller, JHEP **0801**, 063 (2008) [arXiv:0710.5297 [hep-th]].

[10] P. M. Chesler and L. G. Yaffe, Phys. Rev. Lett. **99**, 152001 (2007). [arXiv:0706.0368 [hep-th]]; arXiv:0712.0050 [hep-th] .

[11] S. S. Gubser and S. S. Pufu, Nucl. Phys. B **790**, 42 (2008).

[12] G. T. Bodwin, S. J. Brodsky and G. P. Lepage, Phys. Rev. Lett. **47**, 1799 (1981).

[13] R. Baier, Y. L. Dokshitzer, A. H. Mueller, S. Peigne and D. Schiff, Nucl. Phys. B **484**, 265 (1997) [arXiv:hep-ph/9608322].

[14] S. S. Gubser, Nucl. Phys. B **790**, 175 (2008) [arXiv:hep-th/0612143].

[15] J. Casalderrey-Solana and D. Teaney, JHEP **0704**, 039 (2007) [arXiv:hep-th/0701123];Phys. Rev. D **74**, 085012 (2006) [arXiv:hep-ph/0605199].

[16] Y. Hatta, E. Iancu and A. H. Mueller, arXiv:0803.2481 [hep-th].

[17] B. W. Xiao, arXiv:0804.1343 [hep-th].

[18] R. Baier, Y. L. Dokshitzer, A. H. Mueller, S. Peigne and D. Schiff, Nucl. Phys. B **483**, 291 (1997) [arXiv:hep-ph/9607355].

[19] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)]

[arXiv:hep-th/9711200].

[20] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[21] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[22] C. P. Herzog and D. T. Son, JHEP **0303**, 046 (2003) [arXiv:hep-th/0212072].

[23] D. T. Son and A. O. Starinets, Ann. Rev. Nucl. Part. Sci. **57**, 95 (2007) [arXiv:0704.0240 [hep-th]].

[24] M. Gyulassy and X. N. Wang, Nucl. Phys. B **420**, 583 (1994) [arXiv:nucl-th/9306003].

[25] R. Baier, Y. L. Dokshitzer, S. Peigne and D. Schiff, Phys. Lett. B **345**, 277 (1995) [arXiv:hep-ph/9411409].

[26] J.D. Jackson, "Classical Electrodynamics" 2nd edition, John Wiley and Sons, New York (1975). See especially Eq.(14.28).

[27] A. Mikhailov, arXiv:hep-th/0305196.

[28] A. Yarom, Phys. Rev. D **75**, 105023 (2007) [arXiv:hep-th/0703095].